

Phase transitions and critical behavior of black branes in canonical ensemble

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Abstract

We study the thermodynamics and phase structure of asymptotically flat non-dilatonic as well as dilatonic black branes in a cavity in arbitrary dimensions (D). We consider the canonical ensemble and so the charge inside the cavity and the temperature at the wall are fixed. We analyze the stability of the black brane equilibrium states and derive the phase structures. For the zero charge case we find an analog of Hawking-Page phase transition for these black branes in arbitrary dimensions. When the charge is non-zero, we find that below a critical value of the charge, the phase diagram has a line of first-order phase transition in a certain range of temperatures which ends up at a second order phase transition point (critical point) as the charge attains the critical value. We calculate the critical exponents at that critical point. Although our discussion is mainly concerned with the non-dilatonic branes, we show how it easily carries over to the dilatonic branes as well.

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1 Introduction

Over the years much is known about the thermodynamics and phase structure of black holes in asymptotically AdS space. The reason is that the black holes in AdS space, unlike those in flat space, are thermodynamically stable [1]. Further, more interests were drawn with the advent of AdS/CFT correspondence [2, 3, 4] as the AdS black holes provide a good laboratory for testing the correspondence at finite temperature [5]. So, for example, the AdS black holes are well-known to undergo a Hawking-Page phase transition [1] and by AdS/CFT this corresponds to the confinement-deconfinement phase transition in large N gauge theory [5]. Similarly, the phase structure of the charged AdS black holes which includes, in the canonical ensemble, a similarity with the van der Waals-Maxwell liquid-gas system, can also be understood from the dual field theories [6, 7].

However, it was pointed out [8, 9] that the above mentioned phase structure is not unique for the asymptotically AdS black holes. In fact, very similar phase structure was shown to arise for the suitably stabilized asymptotically flat as well as asymptotically dS black holes [8]. As, the higher dimensional theories like string or M-theory admits higher dimensional black objects like black p -branes [10, 11, 12], it is natural to ask what kind of phase structures do they give rise to – do they have a similar phase structure as the black holes or they have a different phase structure altogether?

Motivated by this we looked into the thermodynamics and the phase structure of black p -brane solution of D -dimensional gravity coupled to $(p+1)$ -form gauge field. In the beginning, we consider only the non-dilatonic branes, (when $D = 11$, they correspond to M2 and M5 branes of M-theory and when $D = 10$, it is the D3-brane of string theory) and then towards the end we show how the analysis carries over to the dilatonic branes as well. The solutions we consider are as usual asymptotically flat and so they are thermodynamically unstable and an isolated black brane would radiate energy in the form of Hawking radiation. In order to restore thermodynamic stability so that equilibrium thermodynamics and the phase structure can be studied, we must consider ensembles that include not only the branes under consideration but also their environment. As self-gravitating systems are spatially inhomogeneous, any specification of such ensembles requires not just thermodynamic quantities of interest but also the place at which they take the specified values. In other words, we place the brane in a cavity *a la* York [13] (see also [14, 15, 16, 17, 18]) and its extension in the charged case [19]. Concretely, we will keep the temperature fixed at the surface of the cavity and as the black p -branes are charged under the $(p+1)$ -form gauge field, we will keep the charge inside the cavity also fixed. This will define a canonical ensemble and we will study the phase structure of the

black p -branes in this ensemble.

After some generalities we first consider the case when the charge in the cavity is fixed to be zero. In this case we find that there is a minimum temperature below which no black brane state exists in equilibrium inside the cavity. But above this temperature there exist two black brane states with different radii. The larger one is locally stable and the smaller one is unstable. The locally stable or ‘supercooled’ large black brane will eventually decay to energetically more favorable state ‘hot flat space’. There is a phase transition temperature at which the ‘hot flat space’ and the corresponding large black brane can coexist⁴. But when the temperature rises above this transition temperature larger black brane becomes globally stable and the ‘hot flat space’ can decay to the black brane. The small black brane still remains unstable. As the temperature rises more, the size of the small black brane decreases and that of the large black brane increases and at infinite temperature the size of the small black brane goes to zero⁵ whereas, the size of the large black brane coincides with the size of the cavity. This situation is analogous to the Hawking-Page transition of AdS black holes or asymptotically flat or dS black holes in a cavity [1, 8].

When the charge is non-zero but fixed inside the cavity, the phase structure becomes more complicated. Here we find that when the charge $|q|$ is greater than a critical value $|q_c|$, there exists a globally stable black brane solution at every temperature in between zero and infinity. However, when $|q| < |q_c|$, there is a range of temperature, where there exist three black brane solutions with different radii. The black branes of the smallest and the largest sizes are locally stable as they correspond to the local minima of the free energy. On the other hand, the intermediate size black brane is unstable, corresponding to the maximum of the free energy, and never exists. The free energies of the largest and the smallest black branes are not the same and one is greater than the other which depends on the temperature of the system. However, there exists a transition temperature, for a given charge $|q| < |q_c|$, where the two free energies become the same. This is the temperature where two black brane states the smallest and the largest can coexist and can make a transition freely from one phase to the other just like the van der Waals-Maxwell liquid-gas phase transition. Since this transition involves an entropy change, therefore, it is a first order phase transition. This structure was also noticed in AdS [6, 7], asymptotically flat and dS black holes [9, 8] in canonical ensemble. When the temperature, in the range

⁴One expects this to be a topological first order phase transition since this involves a topology change as well as an entropy change.

⁵When the temperature is of the order of Planck scale, classical gravity description will break down and one must consider quantum gravity.

we mentioned, is less (more) than the transition temperature, the smallest black brane has lower (higher) free energy than the largest black brane. So, in one case the smallest black brane is globally stable and in other the largest black brane is globally stable. There is a first order phase transition line which depends on the charge $|q| < |q_c|$ and the line gets shrunk as we increase the charge ending up at a second order phase transition point (critical point) when $|q| = |q_c|$. We calculate the critical exponents at this critical point and found that they have very similar structure as in the black hole cases with a universal critical exponent [8, 6] for the specific heat as $-2/3$. Since the stability analysis for the black branes, in general, is quite complicated and extracting exact values of the parameters may not always be easy, we will illustrate the behavior by numerical calculations except for some special cases specified later.

This paper is organized as follows. In section 2, we give the non-dilatonic black p -brane solution in space-time dimensions D and derive the action from which we study the stability and analyze the phase structure. The details of the general stability analysis is discussed in section 3. The phase structure for the case of zero charge in the cavity is considered in section 4. Section 5 discusses the case of non-zero charge. The critical exponents are given in section 6 and we conclude in section 7. The dilatonic brane cases are also mentioned and discussed in appropriate places in the respective section and how the whole analysis of the paper carries over to the dilatonic branes is discussed in the Appendix.

2 Black p -brane solution and the action

The black p -brane solution was originally constructed as a solution to the ten dimensional supergravity containing a metric, a dilaton and a $(p + 1)$ -form gauge field [10]. This was generalized to arbitrary dimensions in [11]. These solutions are given in Lorentzian signature, but for the purpose of studying the thermodynamics [13] we will write the black p -brane solution (without the dilaton field as we will be studying the non-dilatonic branes, the dilatonic branes will be considered in the Appendix) in Euclidean signature

as (see for example [22]),

$$\begin{aligned}
ds^2 &= \Delta_+ \Delta_-^{-\frac{d}{D-2}} dt^2 + \Delta_-^{\frac{\tilde{d}}{D-2}} \sum_{i=1}^{d-1} (dx^i)^2 + (\Delta_+ \Delta_-)^{-1} d\rho^2 + \rho^2 d\Omega_{\tilde{d}+1}^2 \\
A_{[p+1]} &= -i \left[\left(\frac{r_-}{r_+} \right)^{\tilde{d}/2} - \left(\frac{r_- r_+}{\rho^2} \right)^{\tilde{d}/2} \right] dt \wedge dx^1 \cdots \wedge dx^p \\
F_{[p+2]} &\equiv dA_{[p+1]} = -i \tilde{d} \frac{(r_- r_+)^{\tilde{d}/2}}{\rho^{\tilde{d}+1}} d\rho \wedge dt \wedge dx^1 \cdots \wedge dx^p
\end{aligned} \tag{1}$$

In the above we have defined,

$$\Delta_{\pm} = 1 - \left(\frac{r_{\pm}}{\rho} \right)^{\tilde{d}} \tag{2}$$

where r_{\pm} are the two parameters characterizing the solution which are related to the mass and the charge of the black brane. The metric in (1) has an isometry $S^1 \times \text{SO}(d-1) \times \text{SO}(\tilde{d}+2)$ and therefore represents a $(d-1) \equiv p$ -brane in Euclidean signature. The total space-time dimension is $D = d + \tilde{d} + 2$, where the space transverse to the p -brane has the dimensionality $\tilde{d} + 2$. A p -brane couples to the $(p+1)$ -form gauge field whose form and its field-strength are given in (1). It is clear from the Lorentzian form of the above metric that when $r_- = 0$, it reduces to D -dimensional Schwarzschild solution which has an event horizon at $\rho = r_+$, whereas, at $\rho = r_-$, there is a curvature singularity. So, the metric in (1) represents a black p -brane only for $r_+ > r_-$, with $r_+ = r_-$ being its extremal limit. Note that in the above we have defined the gauge potential with a constant shift, following [19], in such a way that it vanishes on the horizon so that it is well-defined on the local inertial frame. For the metric in (1) to be well defined without a conical singularity at $\rho = r_+$, the Euclidean time in the metric is periodic with a periodicity

$$\beta^* = \frac{4\pi r_+}{\tilde{d}} \left(1 - \frac{r_-^{\tilde{d}}}{r_+^{\tilde{d}}} \right)^{\frac{1}{\tilde{d}} - \frac{1}{2}}, \tag{3}$$

which is the inverse of temperature at $\rho = \infty$. The local

$$\beta = \Delta_+^{1/2} \Delta_-^{-\frac{d}{2(D-2)}} \beta^*, \tag{4}$$

which is the inverse of local temperature at ρ when in thermal contact with environment at the same temperature. For the canonical ensemble we have fixed ρ at the wall of cavity denoted by ρ_B , fixed local temperature at ρ_B , fixed local brane volume $V_p = \Delta_-^{\frac{\tilde{d}(d-1)}{2(D-2)}} V_p^*$, with $V_p^* = \int d^p x$ and fixed charge defined as

$$Q_d = \frac{i}{\sqrt{2\kappa}} \int *F_{[p+2]} = \frac{\Omega_{\tilde{d}+1}}{\sqrt{2\kappa}} \tilde{d} (r_+ r_-)^{\tilde{d}/2}. \tag{5}$$

In (5) $\kappa = \sqrt{8\pi G_D}$, where G_D is the D -dimensional Newton's constant, $*F_{[p+2]}$ denotes the Hodge dual for the $(p+2)$ form-field given in (1). Also Ω_n denotes the volume of a unit n -sphere. With these data we will evaluate the action.

The relevant action for the gravity coupled to a $(p+1)$ -form gauge field in a manifold M of dimension D with the Euclidean signature has the form,

$$I_E = I_E(g) + I_E(F) \quad (6)$$

where the first term is the purely gravitational action,

$$I_E(g) = -\frac{1}{2\kappa^2} \int_M d^D x \sqrt{g} R + \frac{1}{\kappa^2} \int_{\partial M} d^{D-1} x \sqrt{\gamma} (K - K_0) \quad (7)$$

consisting of the usual Einstein-Hilbert term and the Gibbons-Hawking boundary term [23] where ∂M denotes the boundary of the manifold M . In the above K is the trace of the extrinsic curvature $K_{\mu\nu}$ defined as,

$$K_{\mu\nu} = -\frac{1}{2} (\nabla_\mu n_\nu + \nabla_\nu n_\mu), \quad \text{and so,} \quad K = -\nabla_\mu n^\mu \quad (8)$$

where n^μ is a space-like vector normal to the boundary and is normalized as $n_\mu n^\mu = 1$. Also $\gamma_{\alpha\beta}$ is the boundary metric with α, β the indices of the boundary coordinates and γ is its determinant. K_0 is the subtraction term which serves as an infra-red regulator so that a finite result can be obtained [1, 13, 19, 3]⁶. This is calculated by embedding the same surface in the flat space.

The second term is due to the form-field $F_{[p+2]}$ and its expression in the canonical ensemble has the form,

$$I_E(F) = \frac{1}{2\kappa^2} \frac{1}{2(d+1)!} \int_M d^D x \sqrt{g} F_{[d+1]}^2 - \frac{1}{2\kappa^2} \frac{1}{d!} \int_{\partial M} d^{D-1} x \sqrt{\gamma} n_\mu F^{\mu\mu_1\cdots\mu_d} A_{\mu_1\cdots\mu_d} \quad (9)$$

Note that for canonical ensemble the charge in the cavity is fixed. For a given gravitational configuration in Euclidean signature, the corresponding thermodynamical partition function in the saddle point (or the zero-loop) approximation can be obtained as $Z \approx e^{-I_E}$,

⁶In the charged case, though this flat space is a solution of the corresponding equations of motion, it is not a thermal state and serves only as an infra-red regulator. One way to see the rationale behind is to first calculate the corresponding action in the grand canonical ensemble for which the flat space is a good reference state subtractor. Then the Euclidean action in the canonical ensemble can be obtained from the calculated Euclidean action in the grand canonical ensemble by a thermodynamical Legendre transformation, noting that the respective action is related to β times the corresponding ensemble potential to leading order. The canonical ensemble action so obtained is also finite and is actually equivalent to that by the subtraction procedure mentioned in the text.

where I_E is evaluated for the given configuration (see for example [20]) and we will evaluate it in the black p -brane configuration given in (1). On the other hand, we also have $Z = e^{-\beta F}$, where β is the inverse temperature of the ensemble and F is the corresponding free energy (for canonical ensemble this is Helmholtz free energy). Therefore, we have in this approximation $F = I_E/\beta$. This is the relevant quantity we are going to evaluate for studying the stability of the black branes and its phase structure.

To evaluate I_E given in (6), we can use the equation of motion of the metric and obtain,

$$R = \frac{\tilde{d} - d}{2(D-2)(d+1)!} F_{[d+1]}^2 \quad (10)$$

Substituting (10) in the action we rewrite I_E as,

$$\begin{aligned} I_E = & \frac{d}{2\kappa^2(D-2)(d+1)!} \int_M d^D x \sqrt{g} F_{[d+1]}^2 + \frac{1}{\kappa^2} \int_{\partial M} d^{D-1} x \sqrt{\gamma} (K - K_0) \\ & - \frac{1}{2\kappa^2} \frac{1}{d!} \int_{\partial M} d^{D-1} x \sqrt{\gamma} n_\mu F^{\mu\mu_1\cdots\mu_d} A_{\mu_1\cdots\mu_d} \end{aligned} \quad (11)$$

In the above we have used $d + \tilde{d} = D - 2$ as mentioned earlier and $d\tilde{d} = 2(D-2)$ for the non-dilatonic branes which is a solution of supergravity with maximal supersymmetry, the type we are considering. We will evaluate each term in (11) separately. For that purpose we need the forms of the normal vector n^μ , the trace of the extrinsic curvatures K and K_0 which can be calculated from the metric in (1) and are given as,

$$\begin{aligned} n^\mu &= (\Delta_+ \Delta_-)^{1/2} \delta_\rho^\mu \\ K &= -\nabla_\mu n^\mu = -\frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} n^\mu) \\ &= -\frac{\tilde{d}+1}{\rho} (\Delta_+ \Delta_-)^{1/2} - \frac{\tilde{d}}{2\rho^{\tilde{d}+1}} \left[\left(\frac{\Delta_-}{\Delta_+} \right)^{1/2} r_+^{\tilde{d}} + \left(\frac{\Delta_+}{\Delta_-} \right)^{1/2} r_-^{\tilde{d}} \right] \\ K_0 &= -\frac{\tilde{d}+1}{\rho} \end{aligned} \quad (12)$$

Now from the metric and the form-field given in (1), it is straightforward to evaluate the action (11) as,

$$I_E = -\frac{\beta V_p \Omega_{\tilde{d}+1}}{2\kappa^2} \rho_B^{\tilde{d}} \left[2 \left(\frac{\Delta_+}{\Delta_-} \right)^{1/2} + \tilde{d} \left(\frac{\Delta_-}{\Delta_+} \right)^{1/2} + \tilde{d} (\Delta_+ \Delta_-)^{1/2} - 2(\tilde{d}+1) \right] \Bigg|_{\rho=\rho_B} \quad (13)$$

Note in the above that ρ is fixed on the cavity as ρ_B . Now since we have $I_E = \beta F$, with $F = E - TS$, the Helmholtz free energy in the canonical ensemble, so this implies that

$I_E = \beta E - S$, with E the energy in the cavity enclosed and S , the entropy for the system. Following Braden et. al. [19] for black hole, we expect that the entropy in the present case (which is just that of the black brane under consideration) should be independent of the choice of the location of the cavity. However, this is not manifest in (13). For this we can rewrite (13) as,

$$I_E = -\frac{\beta V_p \Omega_{\tilde{d}+1}}{2\kappa^2} \rho_B^{\tilde{d}} \left[(\tilde{d}+2) \left(\frac{\Delta_+}{\Delta_-} \right)^{1/2} + \tilde{d}(\Delta_+ \Delta_-)^{1/2} - 2(\tilde{d}+1) \right] \Big|_{\rho=\rho_B} - \frac{4\pi V_p^* \Omega_{\tilde{d}+1}}{2\kappa^2} r_+^{\tilde{d}+1} \left(1 - \frac{r_-^{\tilde{d}}}{r_+^{\tilde{d}}} \right)^{\frac{1}{2} + \frac{1}{\tilde{d}}}, \quad (14)$$

where we have used the expression of β^* given in (3). The last term in (14) is precisely the entropy of the non-dilatonic black p -brane

$$S = \frac{4\pi V_p^* \Omega_{\tilde{d}+1}}{2\kappa^2} r_+^{\tilde{d}+1} \left(1 - \frac{r_-^{\tilde{d}}}{r_+^{\tilde{d}}} \right)^{\frac{1}{2} + \frac{1}{\tilde{d}}} = \frac{4\pi V_p \Omega_{\tilde{d}+1}}{2\kappa^2} r_+^{\tilde{d}+1} \Delta_-^{-\frac{1}{2} - \frac{1}{\tilde{d}}} \left(1 - \frac{r_-^{\tilde{d}}}{r_+^{\tilde{d}}} \right)^{\frac{1}{2} + \frac{1}{\tilde{d}}}, \quad (15)$$

where we have used $V_p = \Delta_-^{\frac{\tilde{d}(d-1)}{2(D-2)}} V_p^*$, and is independent of the location of the cavity. We can also read off the energy for the cavity as

$$E(\rho_B) = -\frac{V_p \Omega_{\tilde{d}+1}}{2\kappa^2} \rho_B^{\tilde{d}} \left[(\tilde{d}+2) \left(\frac{\Delta_+}{\Delta_-} \right)^{1/2} + \tilde{d}(\Delta_+ \Delta_-)^{1/2} - 2(\tilde{d}+1) \right] \Big|_{\rho=\rho_B}, \quad (16)$$

which approaches the ADM mass at $\rho_B \rightarrow \infty$. In casting the Euclidean action in the form of (14), we actually treated the fixed β at the wall of cavity, the inverse of temperature of the environment, to be independent of r_+ for the time being. Then the internal energy E and the entropy S are functions of r_+ only when ρ_B , V_p and Q_d are all fixed as in this ensemble. The stability of the black p -branes can now be discussed from the action (14) when the second line of (15) for entropy is employed to which we turn in the next section.

3 Generalities and stability of black p -branes

We have mentioned that in the canonical ensemble the charge inside the cavity given in (5) is fixed. This implies that r_- is not an independent parameter and can be expressed

in terms of r_+ as,

$$r_- = \left(\frac{\sqrt{2}\kappa Q_d}{\Omega_{\tilde{d}+1} \tilde{d}} \right)^{2/\tilde{d}} \frac{1}{r_+} = \frac{(Q_d^*)^2}{r_+} \quad (17)$$

where we have defined $Q_d^* = [(\sqrt{2}\kappa Q_d)/(\Omega_{\tilde{d}+1} \tilde{d})]^{1/\tilde{d}}$. With these, Δ_- will be given as,

$$\Delta_- = 1 - \left(\frac{(Q_d^*)^2}{r_+ \rho} \right)^{\tilde{d}} \quad (18)$$

We are now considering the canonical ensemble for which the quantities β , V_p , ρ_B and Q_d are all fixed (this implies that β^* , V_p^* are not fixed), and therefore the only parameter which can vary is r_+ . This implies that the entropy will change with r_+ , therefore if we vary the action (14) with respect to r_+ and set that to zero (that is, at the stationary point), we must be able to recover β (since the temperature is a conjugate variable to the entropy) and this in turn will give a non-trivial check that the form of action (14) we have obtained is indeed correct. For this we will replace V_p^* by V_p using the relation, $V_p = \Delta_-^{\frac{1}{2} + \frac{d}{2(D-2)}} V_p^* = \Delta_-^{\frac{1}{2} + \frac{1}{\tilde{d}}} V_p^*$ where we have used $d\tilde{d} = 2(D-2)$ in the second equality and so, (14) takes the form,

$$\begin{aligned} I_E = & -\frac{\beta V_p \Omega_{\tilde{d}+1}}{2\kappa^2} \rho_B^{\tilde{d}} \left[(\tilde{d}+2) \left(\frac{\Delta_+}{\Delta_-} \right)^{1/2} + \tilde{d}(\Delta_+ \Delta_-)^{1/2} - 2(\tilde{d}+1) \right] \\ & - \frac{4\pi V_p \Delta_-^{-\frac{1}{2} - \frac{1}{\tilde{d}}} \Omega_{\tilde{d}+1}}{2\kappa^2} r_+^{\tilde{d}+1} \left(1 - \frac{r_-^{\tilde{d}}}{r_+^{\tilde{d}}} \right)^{\frac{1}{2} + \frac{1}{\tilde{d}}}. \end{aligned} \quad (19)$$

In the expression of Δ_{\pm} here and below the variable ρ must be replaced by constant ρ_B the size of the cavity. From $\partial I_E / \partial r_+ = 0$, we obtain after some simplification,

$$\left[\beta \tilde{d} - 4\pi r_+ \Delta_+^{1/2} \Delta_-^{-\frac{1}{\tilde{d}}} \left(1 - \frac{r_-^{\tilde{d}}}{r_+^{\tilde{d}}} \right)^{\frac{1}{\tilde{d}} - \frac{1}{2}} \right] \left[\tilde{d} + 2 + \left(\frac{\tilde{d}}{2} - \frac{\tilde{d}+2}{2\Delta_-} \right) \left(1 - \frac{r_-^{\tilde{d}}}{r_+^{\tilde{d}}} \right) \right] = 0. \quad (20)$$

Since the second factor in the l.h.s. of (20) is greater than zero, we must have

$$\beta = \frac{4\pi r_+}{\tilde{d}} \Delta_+^{1/2} \Delta_-^{-\frac{1}{\tilde{d}}} \left(1 - \frac{r_-^{\tilde{d}}}{r_+^{\tilde{d}}} \right)^{\frac{1}{\tilde{d}} - \frac{1}{2}} = \Delta_+^{1/2} \Delta_-^{-\frac{d}{2(D-2)}} \beta^*. \quad (21)$$

This is precisely the correct form of β given in (4), which we obtain from the metric in (1). We will use the relation (21) to discuss the stability and the phase structure of black p -branes. For this purpose let us rewrite β explicitly as,

$$\beta = \frac{4\pi r_+}{\tilde{d}} \left(1 - \frac{Q_d^{*2\tilde{d}}}{r_+^{2\tilde{d}}} \right)^{\frac{1}{\tilde{d}} - \frac{1}{2}} \left(1 - \frac{r_+^{\tilde{d}}}{\rho_B^{\tilde{d}}} \right)^{1/2} \left(1 - \frac{Q_d^{*2\tilde{d}}}{r_+^{\tilde{d}} \rho_B^{\tilde{d}}} \right)^{-\frac{1}{\tilde{d}}}, \quad (22)$$

where we have used (17). Now following refs. [19, 8, 9], we define

$$x = \left(\frac{r_+}{\rho_B}\right)^{\tilde{d}} \leq 1, \quad \bar{b} = \frac{\beta}{4\pi\rho_B}, \quad q = \left(\frac{Q_d^*}{\rho_B}\right)^{\tilde{d}}, \quad (23)$$

where the dimensionless parameters \bar{b} , q are fixed⁷ but x is the only parameter which can change. Note that the parameter \bar{b} is related to the inverse of temperature of the environment, q is related to the charge and x is related to the horizon size. Also, as $(Q_d^*)^2/r_+^2 = r_-/r_+ < 1$, $x > q$. In terms of these parameters the above equation of state (or thermal equilibrium condition) for β , (22), can be rewritten as,

$$\bar{b} = b_q(x) \quad (24)$$

where

$$b_q(x) = \frac{1}{\tilde{d}} \frac{x^{1/\tilde{d}}(1-x)^{1/2}}{\left(1 - \frac{q^2}{x^2}\right)^{\frac{1}{2} - \frac{1}{\tilde{d}}} \left(1 - \frac{q^2}{x}\right)^{\frac{1}{\tilde{d}}}}. \quad (25)$$

In the above (25) we have used

$$\Delta_+ = 1 - x, \quad \Delta_- = 1 - \frac{q^2}{x}, \quad 1 - \frac{r_-^{\tilde{d}}}{r_+^{\tilde{d}}} = 1 - \frac{q^2}{x^2}. \quad (26)$$

If we define the reduced Euclidean action as

$$\begin{aligned} \tilde{I}_E &\equiv \frac{2\kappa^2 I_E}{4\pi\rho_B^{\tilde{d}+1} V_p \Omega_{\tilde{d}+1}} \\ &= -\bar{b} \left[(\tilde{d}+2) \left(\frac{1-x}{1-\frac{q^2}{x}} \right)^{1/2} + \tilde{d}(1-x)^{1/2} \left(1 - \frac{q^2}{x} \right)^{1/2} - 2(\tilde{d}+1) \right] \\ &\quad - x^{1+\frac{1}{\tilde{d}}} \left(\frac{1-\frac{q^2}{x^2}}{1-\frac{q^2}{x}} \right)^{\frac{1}{2}+\frac{1}{\tilde{d}}}, \end{aligned} \quad (27)$$

one can show

$$\frac{\partial \tilde{I}_E}{\partial x} = f_q(x) [\bar{b} - b_q(x)] \quad (28)$$

where $b_q(x)$ is as given above in (25) and

$$f_q(x) = (1-x)^{-1/2} \left(1 - \frac{q^2}{x} \right)^{-1/2} \left[\tilde{d}+2 - \frac{\tilde{d}+2}{2} \left(\frac{1-\frac{q^2}{x^2}}{1-\frac{q^2}{x}} \right) + \frac{\tilde{d}}{2} \left(1 - \frac{q^2}{x^2} \right) \right] > 0. \quad (29)$$

⁷Since the charge Q_d as well as Q_d^* denotes the corresponding absolute value, so q also denotes the absolute value. Therefore, from now on, we use q to denote the absolute value of the reduced charge.

The equation of state (24) and the reduced action (27) are the only relevant equations which we need for the discussion of stability and phase structure for the black branes. While we derive them for the non-dilatonic branes, they remain true even for dilatonic branes as we will show in the Appendix. In other words, the analysis given in the following sections works for both the non-dilatonic and dilatonic branes. These two equations depend only on the parameters \bar{b}, q, \tilde{d} and the variable x . The only relevant dimensionality is \tilde{d} which remains unchanged under the so-called double-dimensional reductions [21]. In other words, the black branes related by this reduction will have the same stability and phase structure at least in the approximation employed. For example, the $D = 11$ M2 branes and the $D = 10$ fundamental strings share the same above mentioned properties and so do the other branes related by this kind of reductions (see, for example, [22]).

Now from (28) we find that (at the stationary point)

$$\begin{aligned} \frac{\partial b_q(x)}{\partial x} > 0 & \Rightarrow \frac{\partial^2 \tilde{I}_E}{\partial x^2} < 0, \\ \frac{\partial b_q(x)}{\partial x} < 0 & \Rightarrow \frac{\partial^2 \tilde{I}_E}{\partial x^2} > 0, \end{aligned} \quad (30)$$

Therefore, the system will be stable (at least locally) when $\frac{\partial b_q}{\partial x} < 0$, corresponding to the minimum of the Euclidean action or the free energy. On the other hand, if $\frac{\partial b_q}{\partial x} > 0$, the corresponding Euclidean action is maximum and the free energy is also maximum and the system will be unstable. Since the phase structure for the black p -branes are quite different for the chargeless case and charged case, we will discuss them separately in the next two sections.

4 Chargeless case

In this section we will discuss the stability and the phase structure for the chargeless black p -brane. So, we will put $q = 0$ in all the expressions we obtained in section 3. The expression for the reduced action now takes the form,

$$\tilde{I}_E = -2(\tilde{d} + 1)\bar{b} \left[(1 - x)^{1/2} - 1 \right] - x^{1 + \frac{1}{\tilde{d}}}, \quad (31)$$

whence we have

$$\frac{\partial \tilde{I}_E}{\partial x} = f_0(x) [\bar{b} - b_0(x)]. \quad (32)$$

Where $b_0(x)$ and $f_0(x)$ are given as,

$$\begin{aligned} b_0(x) &= \frac{1}{\tilde{d}} x^{1/\tilde{d}} (1 - x)^{1/2}, \\ f_0(x) &= (\tilde{d} + 1)(1 - x)^{-1/2} > 0. \end{aligned} \quad (33)$$

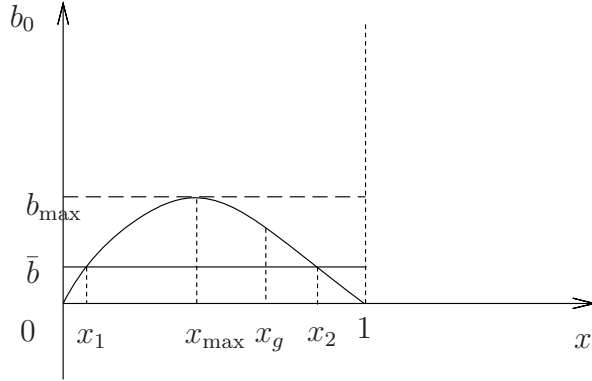


Figure 1: The typical behavior of $b_0(x)$ vs x in the chargeless case.

The equation of state is

$$\bar{b} = b_0(x). \quad (34)$$

At the stationary point of the action, we have

$$\frac{\partial^2 \tilde{I}_E}{\partial x^2} = -f_0(\bar{x}) \left. \frac{\partial b_0(x)}{\partial x} \right|_{x=\bar{x}}, \quad (35)$$

where \bar{x} is a solution of the equation (34). Since $b_0(x) = 0$ at $x = 0, 1$ and $b_0(x) > 0$ between $0 < x < 1$, so $b_0(x)$ has a maximum in between which can be determined from $\partial b_0(x)/\partial x = 0$ and has the value,

$$b_{\max} = \frac{1}{\sqrt{2\tilde{d}}} \left(\frac{2}{\tilde{d}+2} \right)^{\frac{1}{2}+\frac{1}{\tilde{d}}} \Rightarrow T_{\min} = \frac{\sqrt{2\tilde{d}}}{4\pi\rho_B} \left(\frac{\tilde{d}+2}{2} \right)^{\frac{1}{2}+\frac{1}{\tilde{d}}} \quad (36)$$

at

$$x_{\max} = \frac{2}{\tilde{d}+2} \Rightarrow r_{+\max} = \left(\frac{2}{\tilde{d}+2} \right)^{1/\tilde{d}} \rho_B \quad (37)$$

So, there exists a maximum b_{\max} (or minimum temperature T_{\min}) above (or below) which the system can not be in a black brane phase. Now since $\partial \tilde{I}_E/\partial x > 0$ (from (32)) and $I_E = 0$ at $x = 0$ (from (31)), therefore the system favors the ‘hot flat space’. Note that for the four dimensional black holes $D = 4$, and $\tilde{d} = d = 1$, and so, $T_{\min} = 3\sqrt{3}/(8\pi\rho_B)$ and $r_{+\max} = (2/3)\rho_B$ match exactly with the values found in [13, 9]. We will have the same T_{\min} and $r_{+\max}$ for black strings in $D = 5$, black membranes in $D = 6$, up to black D6 branes in $D = 10$ since these branes are related to the four dimensional black hole via the double-dimensional reductions.

For \bar{b} smaller than b_{\max} , we can have two solutions from the equation of state (34), but only the large solution x_2 ($> x_{\max}$) will be locally stable given the relation (35). This behavior of $b_0(x)$ versus x is depicted in Figure 1. However, this does not necessarily imply that the system is in the black brane phase. Only when the local stability becomes a global one, then the system is indeed in the black brane phase. The corresponding x_2 can be determined from requiring the action at the stationary point be negative. The action (31) can be expressed using (34) as,

$$\tilde{I}_E = -\frac{(\tilde{d}+2)\bar{b}}{y} \left(y - \frac{\tilde{d}}{\tilde{d}+2} \right) (y-1), \quad (38)$$

where we have defined

$$y = \sqrt{1-\bar{x}} \quad (39)$$

with $x_2 = \bar{x}$. So, the necessary condition for the global stability can be seen from (38) to be

$$y < \frac{\tilde{d}}{\tilde{d}+2}. \quad (40)$$

This gives

$$\bar{x} > x_g = \frac{4(\tilde{d}+1)}{(\tilde{d}+2)^2} > x_{\max}. \quad (41)$$

Now we find

$$b_g(x = x_g) = \frac{1}{\tilde{d}+2} \left(\frac{4(\tilde{d}+1)}{(\tilde{d}+2)^2} \right)^{1/\tilde{d}} \Rightarrow T_g = \frac{(\tilde{d}+2)}{4\pi\rho_B} \left(\frac{(\tilde{d}+2)^2}{4(\tilde{d}+1)} \right)^{1/\tilde{d}}. \quad (42)$$

So only when $0 < \bar{b} < b_g$ (in this case $x_g < x_2 < 1$), the system is in the black brane phase. On the other hand, for $b_g < \bar{b} < b_{\max}$ (in this case $x_{\max} < x_2 < x_g$), the system, though locally stable, will eventually tunnel to the ‘hot flat space’ at the same temperature. However, at $\bar{b} = b_g$, $\tilde{I}_E = 0$ and so, both the black brane phase with $\bar{x} = x_g$ and the ‘hot flat space’ phase are possible. In other words, this is the place at which the two phases can coexist and the corresponding temperature is the phase transition one. This phase transition is both a topological and a first order one since both the topology and the entropy of the two phases are different before and after the phase transition. Note that when $x \rightarrow 1$, $\bar{b} \rightarrow 0$ as required from (34), but now $\bar{b}/y \rightarrow 1/\tilde{d}$, so the action is still finite and is $\tilde{I}_E = -1$, implying that the system is stable. For the four dimensional black hole we find that the temperature (T_g) where the large black brane becomes globally stable is $27/(32\pi\rho_B)$ which matches with the value found in [13, 9]. Also it is clear from Figure 1 that, as $b_0(x)$ decreases or the temperature increases the size of the small black

brane decreases and that of the large black brane increases and eventually when $b_0 \rightarrow 0$ or $T \rightarrow \infty$, the size of the small black brane goes to zero and the size of the large black brane approaches the size of the cavity. The phase transition we found in the present case is analogous to the Hawking-Page transition for the AdS black holes.

5 Charged case

In this section we will study the stability and phase structure of black p -brane in the more general case where the charge enclosed by the cavity is non-zero and fixed. In section 3, while discussing the generalities for the stability of black p -brane we found that the system will be stable when $\partial b_q(x)/\partial x < 0$ and will be unstable when $\partial b_q(x)/\partial x > 0$. We will show that there exists a critical charge q_c , such that when $q > q_c$, the system will be globally stable as $\partial b_q(x)/\partial x$ is always less than zero (see Figure 3 below), but when $q < q_c$, $\partial b_q(x)/\partial x > 0$ in some region as shown in Figure 2. We mentioned in section 3 that the parameter x lies between 1 and q . Since $b_q(x) \rightarrow \infty$ as $x \rightarrow q$ and $b_q(x) \rightarrow 0$ as $x \rightarrow 1$ (see eq.(25)), therefore, when $q < q_c$, $b_q(x)$ does not decrease monotonically (as seen from Figure 2) and $\partial b_q(x)/\partial x > 0$ in some region of $q < x < 1$. In other words, we should have a minimum of $b_q(x)$ (b_{\min}) occurring at $x = x_{\min}$ and a maximum of $b_q(x)$ (b_{\max}) occurring at $x = x_{\max}$. When $b_{\min} < b_q(x) < b_{\max}$ and $x_{\min} < x < x_{\max}$, $\partial b_q(x)/\partial x > 0$ and the system is unstable in this region and there exist no stable brane phases. On the other hand, when $b_q(x) > b_{\max}$ or $b_q(x) < b_{\min}$, $\partial b_q(x)/\partial x < 0$ and the system is stable (at least locally). So, for a given \bar{b} with $b_{\min} < \bar{b} < b_{\max}$, there will be three solutions to the equation of state (24). If we denote the three solutions as x_1 , x_2 and x_3 with $x_1 < x_2 < x_3$, then x_1 and x_3 correspond to the local minima of the free energy and x_2 corresponds to the maximum. Among the two minima, one expects that the one with the lower free energy will be globally stable and a transition will occur from the state of higher free energy to the lower free energy. So, it is important to find which of the two black branes have lower free energy. To determine this we write from (28),

$$\tilde{I}_E(x_3) - \tilde{I}_E(x_1) = S(x_2, x_1) - \bar{S}(x_3, x_2), \quad (43)$$

where

$$S(x_2, x_1) = \int_{x_1}^{x_2} dx f_q(x) [\bar{b} - b_q(x)] \geq 0, \quad (44)$$

$$\bar{S}(x_3, x_2) = \int_{x_2}^{x_3} dx f_q(x) [b_q(x) - \bar{b}] \geq 0, \quad (45)$$

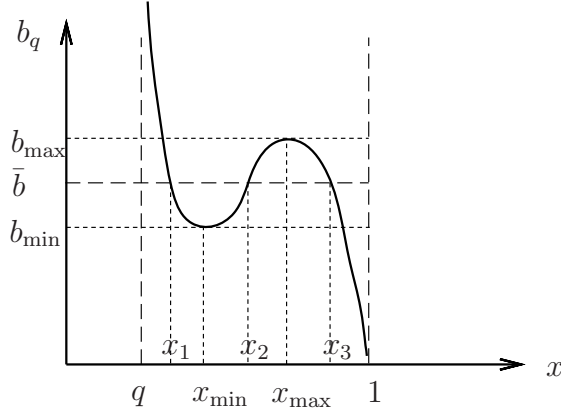


Figure 2: The typical behavior of $b_q(x)$ vs x when there is a phase transition ($q < q_c$).

with $b_q(x)$ as given in (25). Note that for $\bar{b} = b_{\min}$, the points x_2 and x_1 coincide and so $S(x_2, x_1) = 0$ and $\bar{S}(x_2, x_3)$ takes a maximum value. Similarly, for $\bar{b} = b_{\max}$, the points x_2 and x_3 coincide and therefore, $\bar{S}(x_2, x_3) = 0$ and $S(x_2, x_1)$ takes a maximum value. Thus both the function $S(x_2, x_1)$ and $\bar{S}(x_3, x_2)$ change continuously from 0 to their maximum value as \bar{b} is varied from b_{\min} to b_{\max} . So, in between there must exist a fixed value of \bar{b} which we call \bar{b}_t (inverse of which is related to a phase transition temperature) for each given charge $q < q_c$, for which $S(x_2, x_1) = \bar{S}(x_3, x_2)$ and therefore the Euclidean action or the free energies of the two stable black brane configurations of sizes x_1 and x_3 are the same. In other words, this is a phase transition temperature where the two black brane phases coexist and make a transition freely from one phase to the other, much like a van der Waals-Maxwell liquid-gas phase transition. This was also noticed earlier for asymptotically AdS, dS and flat black holes in canonical ensemble [6, 9, 8].

Now since at b_{\max} , $\bar{S}(x_3, x_2) = 0$ and $S(x_2, x_1)$ is maximum and at \bar{b}_t , they are the same, so, if \bar{b} lies in between i.e., $b_{\max} > \bar{b} > \bar{b}_t$, $S(x_2, x_1) > \bar{S}(x_3, x_2)$, or $\tilde{I}_E(x_3) > \tilde{I}_E(x_1)$. So, in this case the small black brane phase is globally stable. Similarly, since at b_{\min} , $S(x_2, x_1) = 0$ and $\bar{S}(x_3, x_2)$ is maximum and at \bar{b}_t , they are the same, so, if \bar{b} lies in between i.e., $b_{\min} < \bar{b} < \bar{b}_t$, $\bar{S}(x_3, x_2) > S(x_2, x_1)$, or $\tilde{I}_E(x_1) > \tilde{I}_E(x_3)$. So, in this case large black brane phase is globally stable. Thus we conclude that if the temperature is above the transition value, but below T_{\max} , large black brane phase is globally stable and if it is below the transition value, but above T_{\min} , the small black brane phase is globally stable. So, there will be a phase transition from the small black brane to large black brane or *vice versa* depending on whether the temperature of the black brane is below or above the transition temperature.

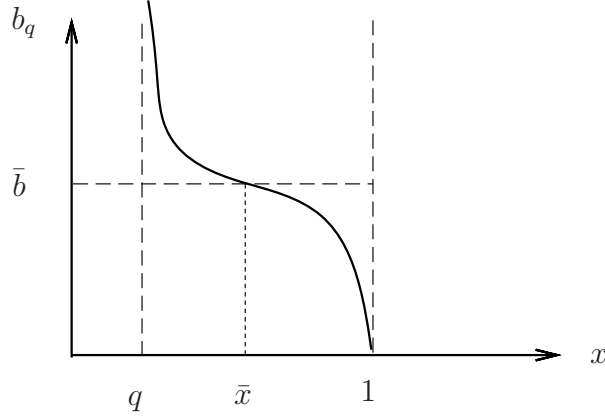


Figure 3: $b_q(x)$ decreases monotonically with x when $q < x < 1$ ($q > q_c$).

It is clear from the expression of entropy given in the last term of (14) that for given $q < q_c$, the entropy will depend on the parameter r_+ or x and so, the entropy will change during the phase transition we just mentioned. Therefore, this is a first order phase transition for which the entropy has a discontinuity. In fact there is a first order phase transition line when we move the charge $q < q_c$ towards $q = q_c$ and the line gets shrunk ending up at a second order phase transition point (critical point) for $q = q_c$, which occurs at $x = x_c$, and where the entropy discontinuity disappears.

Having understood the qualitative features of the equilibria and the phase structure of black p -branes with non-zero charge inside the cavity, we will try to understand the structure in a more quantitative way and then corroborate our observations by numerical calculations how the various situations for $q > q_c$, $q = q_c$ and $q < q_c$ described here arise in three different values of \tilde{d} , for examples, for the non-dilatonic branes (we will consider only $\tilde{d} = 3$ in $D = 11$, which corresponds to M5-brane, $\tilde{d} = 6$ in $D = 11$, which corresponds to M2-brane and $\tilde{d} = 4$ in $D = 10$, which corresponds to D3-brane) though these calculations work for the corresponding dilatonic branes related via the double-dimensional reductions in other dimensions as well.

For understanding the stability, as we mentioned, the quantity to look at is $\partial b_q(x)/\partial x$. From (25) we find,

$$\frac{\partial b_q(x)}{\partial x} = - \frac{x^{1/\tilde{d}} \left\{ \left(1 + \frac{\tilde{d}}{2}\right)x^4 - \left[1 + \left(2 + \frac{\tilde{d}}{2}\right)q^2\right]x^3 - 3q^2\left(\frac{\tilde{d}}{2} - 1\right)x^2 + q^2[\tilde{d} - 1 + \frac{3\tilde{d}}{2}q^2]x - \tilde{d}q^4 \right\}}{\tilde{d}^2 x^4 (1-x)^{1/2} \left(1 - \frac{q^2}{x^2}\right)^{1+\frac{\tilde{d}}{2(D-2)}} \left(1 - \frac{q^2}{x}\right)^{1+\frac{\tilde{d}}{2(D-2)}}}. \quad (46)$$

The position of the extremality will be determined from the vanishing of the numerator of the above equation (46), i.e.,

$$\left(1 + \frac{\tilde{d}}{2}\right) x^4 - \left[1 + \left(2 + \frac{\tilde{d}}{2}\right) q^2\right] x^3 - 3q^2 \left(\frac{\tilde{d}}{2} - 1\right) x^2 + q^2 \left[\tilde{d} - 1 + \frac{3\tilde{d}}{2} q^2\right] x - \tilde{d}q^4 = 0, \quad (47)$$

This is a quartic equation and has four roots in general. We will make some observation about the roots of this equation which will support the various structures we described qualitatively in this section and then give some numerical solution of this equation in some special cases. First note that the discriminant of the above equation (47) which tells us about the roots has the form,

$$\Delta(q, \tilde{d}) = -\frac{(q^2 - 1)^3 q^6}{16} \left[(4(\tilde{d} - 1) - 3\tilde{d}(4 + \tilde{d})q^2)^3 - 108\tilde{d}^2(2 + \tilde{d} - \tilde{d}^2)^2 q^2(1 - q^2) \right] \quad (48)$$

The discriminant will vanish within $0 < q^2 < 1$, if the last factor within the square bracket in (48) vanishes in that range. This will be determined by the intersection point of the first term, i.e., a cubic curve with the second term, i.e., a parabola. The parabola meets the q^2 -axis at $q^2 = 0$ and at $q^2 = 1$ and remains positive in this range. On the other hand the cubic curve takes a positive value of $4^3(\tilde{d} - 1)^3$ for $\tilde{d} > 1$ at $q^2 = 0$, monotonically decreases and meets the q^2 -axis at $q_0^2 = 4(\tilde{d} - 1)/(3\tilde{d}(4 + \tilde{d})) < 1$ (for $\tilde{d} > 1$)⁸. Therefore, there is a unique crossing point q_c^2 of the cubic curve and the parabola in the range $0 < q^2 < 1$ with $0 < q_c^2 < q_0^2 = 4(\tilde{d} - 1)/(3\tilde{d}(4 + \tilde{d})) < 1$. This shows the existence of a unique critical point at $q = q_c$, where $\partial b_q(x)/\partial x$ vanishes. Note that since this is a single extremum, as $b_q(x)$ varies from ∞ to 0, this can not be a maximum or minimum, but is an inflection point where $\partial^2 b_q(x)/\partial x^2$ also vanishes. This feature is reflected in Figure 4 below. Note that for $\tilde{d} = 1$, $q_0^2 = 0$ and the intersection now occurs at $q_0 = 0$ which is not in the range of $0 < q^2 < 1$, therefore we don't expect a critical point and further a possible phase transition to occur. This will be checked explicitly in a subsection of this section later.

When $q^2 > q_c^2$, we have $\Delta(q, \tilde{d}) < 0$, so (47) (since this is a quartic equation, there are four roots of this equation in general), must have a pair of complex conjugate roots. Also since the ratio of the coefficient of x^4 term and the constant term (it is $-\tilde{d}q^4/(1 + \tilde{d}/2)$) is negative, implying that the product of the four roots are negative, so one of the roots must be negative. Hence, there can be at most one root in the region $q < x < 1$. However, for $\tilde{d} > 2$, as the l.h.s. of equation (47) is positive at both $x = q$ (it is $(-2 + \tilde{d})(-1 + q^2)q^3$) and $x = 1$ (it is $\tilde{d}(-1 + q^2)^2/2$), the number of roots, if existing at all in the region

⁸We will discuss $\tilde{d} = 1$ case separately in a subsection of this section.

$q < x < 1$, must be even⁹. So, there can not be any root of (47) in the range $q < x < 1$ if $q^2 > q_c^2$ and this is consistent with the typical behavior given in Figure 3.

When $q^2 < q_c^2$, we have $\Delta(q, \tilde{d}) > 0$ and this implies that (47) has either two pairs of complex conjugate roots or all real roots. Also since the product of all four roots must be negative, then the roots must all be real and the number of negative roots must be odd (one or three). By the argument of previous paragraph, the number of roots, if existing at all in the range $q < x < 1$, must be even (two or none). The finding that x_c falling in the region $q < x < 1$ for $q = q_c$, for each of the cases considered in the following, implies that indeed there exist two roots in the region $q < x < 1$ for $q^2 < q_c^2$. This is also consistent with the typical behavior given in Figure 2.

Now we give some numerical calculation to illustrate the above picture. When $\Delta(q, \tilde{d}) = 0$, one expects that q^2 in (48) has only one real positive root q_c^2 . For this critical q_c , we should have $x_{\min} = x_{\max}$ in the region $q_c < x < 1$. Let us examine carefully if this is indeed true. Let us first consider M5-brane, i.e., $\tilde{d} = 3$ first. Now $\Delta(q_c, \tilde{d} = 3) = 0$ gives us,

$$-512 + 27648y - 110808y^2 + 250047y^3 = 0, \quad (49)$$

where we have defined $y = q_c^2$. One can check that indeed y has only one positive real solution given by

$$y = \frac{4 \left(114 - 4478 \left(\frac{2}{-450413 + 222607\sqrt{5}} \right)^{1/3} + 2^{2/3} (-450413 + 222607\sqrt{5})^{1/3} \right)}{3087} \approx 0.020058, \quad (50)$$

whence we get

$$q_c = 0.141626. \quad (51)$$

Substituting this in (47) we obtain

$$-0.00120697 + 0.0419264x - 0.030087x^2 - 1.0702x^3 + \frac{5}{2}x^4 = 0, \quad (52)$$

which indeed gives a unique solution in the region $q_c < x < 1$ as,

$$x_c = 0.292675 \quad (53)$$

(the other two irrelevant solutions are $x = -0.187352$ and $x = 0.030083$). So, we now have the critical b_c as,

$$b_c = \frac{x_c^{1/3}(1 - x_c)^{1/2}}{3(1 - \frac{q_c^2}{x_c^2})^{1/6}(1 - \frac{q_c^2}{x_c})^{1/3}} = 0.199253. \quad (54)$$

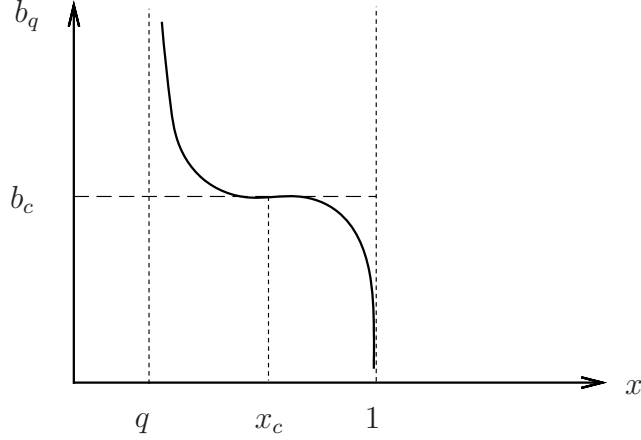


Figure 4: The typical behavior of $b_q(x)$ vs x when there is a turning point for which $x_{\min} = x_{\max} = x_c$ ($q = q_c$).

From (47), one expects that there exist a minimum of $b_q(x)$, occurring at $x = x_{\min}$ and a maximum, occurring at $x = x_{\max}$, when $q < q_c$, corresponding to Figure 2 ($\Delta(q, \tilde{d}) > 0$ in this case), while such x_{\min} and x_{\max} should not exist when $q > q_c$ corresponding to Figure 3 ($\Delta(q, \tilde{d}) < 0$ in this case). Let us take two explicit examples, one for each case, showing that this is indeed true. When we take $q = 0.150000 > q_c$ which is slightly larger than the critical value, we find indeed that there exist no real solution in the region $q < x < 1$. While if we take $q = 0.130000 < q_c$ which is slightly smaller than the critical value, we find that indeed there exist two solutions in the region $q < x < 1$, one is $x_{\min} = 0.229179$ and the other is $x_{\max} = 0.341762$, as expected.

For M2-brane, $\tilde{d} = 6$, we find exactly the same behavior. In this case $\Delta(q_c, \tilde{d} = 6) = 0$ gives (see (48))

$$-8000 + 3264192y - 4992192y^2 + 5832000y^3 = 0, \quad (55)$$

where again we have defined $y = q_c^2$. The only real positive solution of (55) gives,

$$y = \frac{321 - 9506 \left(\frac{7}{-203167 + 168250\sqrt{2}} \right)^{1/3} + 2 \cdot 7^{2/3} (-203167 + 168250\sqrt{2})^{1/3}}{1125},$$

$$\approx 0.00246007, \quad (56)$$

which gives

$$q_c = 0.049599. \quad (57)$$

⁹We will discuss $\tilde{d} = 2$ in a separate subsection of this section.

This in turn gives the unique solution

$$x_c = 0.175176, \quad (58)$$

from the corresponding equation (47)

$$-0.0000363115 + 0.0123548x - 0.0147604x^2 - 1.0123x^3 + 4x^4 = 0, \quad (59)$$

in the region of $q_c < x < 1$. We also have

$$b_c = 0.116698. \quad (60)$$

The other cases $q > q_c$ and $q < q_c$ can be discussed similarly as above.

For D3-brane, $\tilde{d} = 4$, $\Delta(q_c, \tilde{d} = 4) = 0$ gives

$$-1728 + 214272y - 504576y^2 + 884736y^3 = 0, \quad (61)$$

whose unique positive real solution is

$$y = \frac{73 - 1315 \left(\frac{5}{-34367 + 16512\sqrt{6}} \right)^{1/3} + 5^{2/3} (-34367 + 16512\sqrt{6})^{1/3}}{384},$$

$$\approx 0.00822139, \quad (62)$$

which gives

$$q_c = 0.090672. \quad (63)$$

With this critical q_c , we have the unique

$$x_c = 0.238800 \quad (64)$$

in the region of $q_c < x < 1$ from the corresponding equation (47)

$$-0.000270365 + 0.0250697x - 0.0246642x^2 - 1.03289x^3 + 3x^4 = 0. \quad (65)$$

Now

$$b_c = 0.159921. \quad (66)$$

Once again, the other cases $q > q_c$ and $q < q_c$ can be similarly discussed.

We have given the numerical results for $\tilde{d} = 3, 6$, and 4 since they are related to the M5, M2 and D3 branes. As indicated earlier and with the discussion given in the Appendix for dilatonic branes, any brane, non-dilatonic or dilatonic, related to each of these branes by the so-called double-dimensional reductions will share the same properties,

since the only dimensionality entering this discussion is \tilde{d} and it remains the same under this reduction.

Note that since we have $1 \leq \tilde{d} \leq 7$, for completeness and to show that the critical value x_c always falls in the range $q_c < x_c < 1$, we give the results also for $\tilde{d} = 5$ and 7 ($\tilde{d} = 1, 2$ will be discussed separately). The critical values for $\tilde{d} = 5$ are ($q_c = 0.064944$, $x_c = 0.202012$, $b_c = 0.134632$) and for $\tilde{d} = 7$ they are ($q_c = 0.039529$, $x_c = 0.154691$, $b_c = 0.103210$). We collect the relevant quantities for the cases $3 \leq \tilde{d} \leq 7$ in the tabular form as,

\tilde{d}	q_c	x_c	b_c
3	0.141626	0.292656	0.199253
4	0.090672	0.238800	0.159921
5	0.064944	0.202012	0.134632
6	0.049599	0.175176	0.116698
7	0.039529	0.154691	0.103210

The table above shows that the critical size of the black brane always lies in the range $q_c < x_c < 1$, where q_c is related to the absolute value of critical charge and b_c is related to the inverse critical temperature as defined earlier. We observe that the critical quantities all decrease as \tilde{d} increases.

5.1 $\tilde{d} = 2$

The general equilibria and the phase structure that we discussed in this section does not apply to $\tilde{d} = 2, 1$ cases as we mentioned before and so, we will study these two cases separately here in the next two subsections. Let us first discuss $\tilde{d} = 2$ case. We find from (25) that for $\tilde{d} = 2$

$$b_q(x) = \frac{x^{1/2}(1-x)^{1/2}}{2\left(1 - \frac{q^2}{x}\right)^{1/2}}. \quad (67)$$

The corresponding equation (from (47)) for finding the extrema for $\tilde{d} = 2$ is

$$2x^4 - (1 + 3q^2)x^3 + q^2(1 + 3q^2)x - 2q^4 = 0 \quad (68)$$

which can be factorized as,

$$(x^2 - q^2)(x - x_+)(x - x_-) = 0 \quad (69)$$

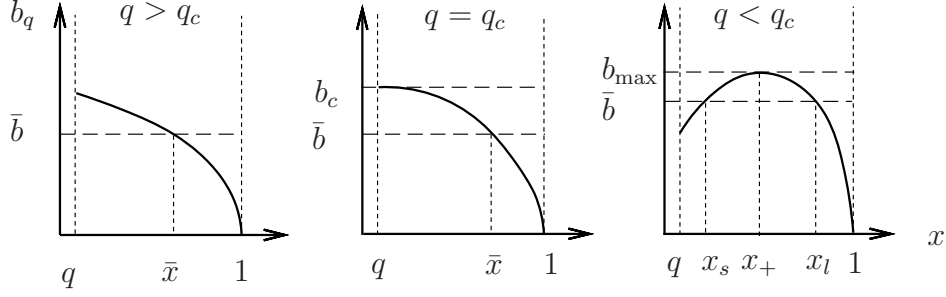


Figure 5: The typical behaviors of $b_q(x)$ vs x for $\tilde{d} = 2$.

where,

$$x_{\pm} = \frac{1}{4} \left(1 + 3q^2 \pm \sqrt{\tilde{\Delta}} \right) \quad (70)$$

with $\tilde{\Delta} = (1 - q^2)(1 - 9q^2)$. The $x^2 = q^2$ solutions are irrelevant here and x_{\pm} can be real only if $\tilde{\Delta} \geq 0$ implying $0 < q \leq 1/3$. Note that the discriminant for the present case from (48) has the form,

$$\Delta(q, 2) = 4q^6(1 - q^2)^3(1 - 9q^2)^3 \quad (71)$$

This gives the same requirement as $\tilde{\Delta}$ for the reality of the solutions. Comparing with our earlier discussion of $\Delta(q, \tilde{d})$, we note that here there is no parabola and the cubic curve meets q^2 axis at $q_0 = 1/3$. So, this is also the critical point $q_c = q_0 = 1/3$ which is consistent with the relevant solution of $\tilde{\Delta} = 0$. Further we note that for this case, $b_q(x = q) = \sqrt{q}/2$ which is different from ∞ (for non-zero charge), the value for $\tilde{d} > 2$, but the other end $b(1) = 0$ remains the same.

Now let us see the phase structure in detail (see Figure 5). When $1 > q > q_c = 1/3$, we see that there exists no extrema for $q < x < 1$, since now $\tilde{\Delta} < 0$ and

$$\frac{\partial b_q(x)}{\partial x} = -\frac{b_q(x)}{2} \left[\frac{2x^2 - x(1 + 3q^2) + 2q^2}{x(1 - x)(x - q^2)} \right] < 0. \quad (72)$$

So, for now, when $0 < \bar{b} < b_q(q)$, we have a stable black brane¹⁰. For $\bar{b} > b_q(q)$, there is no stable black brane and we don't have a description available for such phase.

For $q = q_c = 1/3$, $\tilde{\Delta}$ vanishes and the two roots x_{\pm} are equal and has the value $1/3 = q_c$. So, $b_q(x = x_c) = b_c$ is a fake critical point which is not accessible since this is also an extremal point. This point is only marginally stable so long as the thermodynamics is

¹⁰Unless we have a phase like 'hot flat space', but now carrying a charge, whose free energy can be the lowest to be the globally stable phase, this black brane phase is globally stable.

concerned. Now as we increase x beyond this value, $b_q(x)$ monotonically decreases and goes to zero at $x = 1$ and in this range $\partial b_q/\partial x < 0$. Thus we find that in the range $b_c = 1/(2\sqrt{3}) > \bar{b} > b_q(1) = 0$, there exists a stable black brane phase. However, if $\bar{b} > 1/(2\sqrt{3})$ there are no black brane phase or such description is not available. In terms of temperature, if the temperature T is below $\sqrt{3}/(2\pi\rho_B)$, there are no black brane phase, however, if the temperature is in the range $\sqrt{3}/(2\pi\rho_B) < T < \infty$, there is a stable black brane phase.

For $q < q_c = 1/3$, $\tilde{\Delta} > 0$, so in this case we have two real solutions of $\partial b_q(x)/\partial x = 0$ given by (70). One can check that $x_- < q$ and $q < x_+ < 1$ and so only x_+ is the relevant solution which gives a maximum of $b_q(x)$ at $x = x_+$ as

$$b_{\max} = \frac{\left(1 + 3q^2 + \sqrt{(1 - q^2)(1 - 9q^2)}\right)^{1/2} \left(3\sqrt{1 - q^2} - \sqrt{1 - 9q^2}\right)}{8\sqrt{2}} > b_q(q) = \frac{\sqrt{q}}{2}. \quad (73)$$

Note that for $q < x < x_+$, $\partial b_q(x)/\partial x > 0$ while for $x_+ < x < 1$, it is less than zero. Therefore in the range $0 < \bar{b} < b_q(q) = \sqrt{q}/2$, there is only one black brane phase with $1 > x > x_+$ which is stable (we assume that there exists no other stable phase) and in the range $b_q(q) < \bar{b} < b_{\max}$, there are two black brane phases in which the smaller one is unstable and the larger one is stable. For $\bar{b} > b_{\max}$ there is no black brane phase or such a description is not available.

5.2 $\tilde{d} = 1$

For $\tilde{d} = 1$, we find from (25) the form of the parameter $b_q(x)$ as

$$b_q(x) = \frac{x(1 - x)^{1/2} \left(1 - \frac{q^2}{x^2}\right)^{\frac{1}{2}}}{1 - \frac{q^2}{x}}. \quad (74)$$

From $b_q(x = q) = b_q(x = 1) = 0$ and the fact that $b_q(x) > 0$ for $q < x < 1$, there must exist one and only one extremum which corresponds to a maximum, denoted as b_{\max} (see Figure 6). This is due to the absence of a critical point as discussed earlier for the present case. Also as discussed in the previous section, since now $q_0^2 = 0$, so

$$\Delta(q, 1) = \frac{(15)^3(q^2 - 1)^3 q^8}{16} \left[q^4 - \frac{16}{125} q^2 + \frac{16}{125} \right] < 0, \quad (75)$$

for $0 < q^2 < 1$. So there must exist a pair of complex conjugate solutions for the following extremal equation of $b_q(x)$ which is a special case of (47) for $\tilde{d} = 1$,

$$\frac{3}{2}x^4 - \left(1 + \frac{5}{2}q^2\right)x^3 + \frac{3}{2}q^2x^2 + \frac{3}{2}q^4x - q^4 = 0. \quad (76)$$

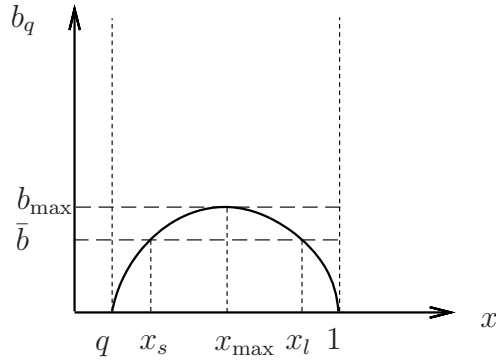


Figure 6: The typical behavior of $b_q(x)$ vs x for $\tilde{d} = 1$.

Furthermore, as the product of four roots of the above equation is less than zero, this implies that the other two roots must be one negative and one positive. Given that there exists a maximum, this positive root must be the one we expect and should fall in the region $q < x_{\max} < 1$. Let us take a special case of $q = 0.50$ as an example. Now the four solutions of (76) are

$$x_1 = -0.30, \quad x_2 = 0.33 - 0.29i, \quad x_3 = 0.33 + 0.29i, \quad x_4 = 0.73, \quad (77)$$

which are exactly as expected with only $x_4 = 0.73 > q$ as the location of maximum of $b_q(x)$ for this $q = 0.50$. The situation here is similar to the chargeless case but we have here $q < x < 1$ for $b_q(x)$ vs x . In addition, we do not have the corresponding transition to ‘hot flat space’, i.e., the Hawking-Page transition, since the charge is fixed for the present system.

For a given $0 < \bar{b} < b_{\max}$, there are two blackbrane solutions: the small black brane is unstable while the large black brane is at least locally stable. Unless some new phase carrying the same charge is found with lower free energy, we can not justify whether such a large brane is globally stable or locally stable.

This therefore completes our analysis of the equilibria and the phase structure of black p -branes in the case of non-zero charge. We have seen that the $\tilde{d} = 2$ serves as a borderline which distinguishes the $\tilde{d} = 1$ case from the $\tilde{d} > 2$ cases. For the former case, the introduction of charge doesn’t change qualitatively the stability behavior except for putting a new lower bound on the x set by the charge. In addition, there doesn’t appear to exist the obvious analog of Hawking-Page type transition. Apart from this, the phase structure remains basically the same as the chargeless case. For the latter case, however, the introduction of charge does significantly change the stability as well as the

phase structure from the chargeless case as described in detail in this section. One of striking features is the appearance of a critical charge which determines both the stability behavior and the phase structure of the underlying system. When the charge is less than the critical charge, there exists a first order phase transition line which ends at a second order phase transition point which is the critical point at the critical charge. We would like to remark that although the details of the phase structure is quite different for $\tilde{d} > 2$ for black branes, the qualitative structure is very similar to those of asymptotically AdS, dS and flat black holes in canonical ensemble studied earlier[6, 8, 9].

6 Critical exponents

We observed in section 5 for the case of charged black branes, that when the charge is below certain critical value, $q < q_c$, there exists two stable black brane states in certain range of temperature. Let us denote the sizes of the two black branes as x_s and x_ℓ , where the former denotes the small black brane and the latter is the large black brane. There is a transition temperature where the free energies of the two black branes are the same and at this temperature these two black brane phases coexist. This transition temperature, which is completely determined by the given charge $q < q_c$, therefore, forming a first order phase transition line, can be described by $T_t(q)$ (with the subscript ‘t’ denoting it as a phase transition temperature). As the charge increases, this phase transition line ends up to a second order phase transition point (critical point) at $q = q_c$. We would like to calculate the critical exponents at this critical point. Expanding $b_q(x)$ around the critical point x_c we have,

$$b_q - b_c = \frac{1}{3!} \left. \frac{\partial^3 b_q}{\partial x^3} \right|_{x=x_c} (x - x_c)^3 + \dots \quad (78)$$

Note that at the critical point the first and the second order derivatives of $b_q(x)$ with respect to x are zero. From (25) we find,

$$\left. \frac{\partial^3 b_q}{\partial x^3} \right|_{x=x_c} = - \frac{6 x_c^{\frac{1}{\tilde{d}}} \left\{ 2 \left(1 + \frac{\tilde{d}}{2} \right) x_c^2 - \left[1 + \left(2 + \frac{\tilde{d}}{2} \right) q_c^2 \right] - q_c^2 \left(\frac{\tilde{d}}{2} - 1 \right) \right\}}{\tilde{d}^2 x_c^4 (1 - x_c)^{\frac{1}{2}} \left(1 - \frac{q_c^2}{x_c^2} \right)^{\frac{\tilde{d}-2}{2\tilde{d}}+1} \left(1 - \frac{q_c^2}{x_c} \right)^{\frac{1}{\tilde{d}}+1}}, \quad (79)$$

where we have used

$$\left. \frac{\partial b_q}{\partial x} \right|_{x=x_c} = 0, \quad \left. \frac{\partial^2 b_q}{\partial x^2} \right|_{x=x_c} = 0. \quad (80)$$

We mentioned in section 2, that the form of the entropy can be read off from the last term of the Euclidean action (14). If we now define the reduced entropy for fixed ρ_B and

V_p as, $\tilde{S} = 2\kappa^2 S/(\Omega_{\tilde{d}+1} V_p \rho_B^{\tilde{d}+1})$, then from (27) we read off the form of reduced entropy as,

$$\tilde{S} = 4\pi x^{\frac{\tilde{d}+1}{d}} \left(1 - \frac{q^2}{x}\right)^{-\frac{1}{2}-\frac{1}{d}} \left(1 - \frac{q^2}{x^2}\right)^{\frac{1}{2}+\frac{1}{d}}. \quad (81)$$

We now expand the entropy around the critical point as,

$$\tilde{S} - \tilde{S}_c = \left. \frac{\partial \tilde{S}}{\partial x} \right|_{x=x_c} (x - x_c) + \dots, \quad (82)$$

where from (81),

$$\frac{\partial \tilde{S}}{\partial x} = \frac{2\pi x^{\frac{1}{d}} \left[2(\tilde{d}+1)x^3 - (3\tilde{d}+4)q^2x^2 + 2q^2x + q^4\tilde{d} \right]}{\tilde{d}x^3 \left(1 - \frac{q^2}{x}\right)^{\frac{3}{2}+\frac{1}{d}} \left(1 - \frac{q^2}{x^2}\right)^{\frac{\tilde{d}-2}{2d}}}. \quad (83)$$

We would like to have the expansion of entropy not around the x_c , rather around the reduced critical temperature $\tau_c = 1/b_c$. Note that near the critical temperature,

$$\begin{aligned} \tau - \tau_c &= \frac{1}{b_q} - \frac{1}{b_c} \\ &= -\frac{1}{b_c^2} \frac{1}{3!} \left. \frac{\partial^3 b_q}{\partial x^3} \right|_{x=x_c} (x - x_c)^3 + \dots \end{aligned} \quad (84)$$

where we have used (78). So, using (84) we can write (82) in terms of the reduced temperature as,

$$\tilde{S} - \tilde{S}_c = \left. \frac{\partial \tilde{S}}{\partial x} \right|_{x=x_c} \left[\frac{b_c^2}{-\frac{1}{3!} \left. \frac{\partial^3 b_q}{\partial x^3} \right|_{x=x_c}} \right]^{1/3} (\tau - \tau_c)^{1/3} + \dots \quad (85)$$

The reduced specific heat therefore can be calculated as,

$$\begin{aligned} \tilde{c}_v &= T \frac{\partial \tilde{S}}{\partial T} = \tau \frac{\partial \tilde{S}}{\partial \tau} \\ &= \frac{1}{3} \left. \frac{\partial \tilde{S}}{\partial x} \right|_{x=x_c} \left[\frac{1}{-\frac{1}{3!} \left. \frac{\partial^3 b_q}{\partial x^3} \right|_{x=x_c} b_c} \right]^{1/3} (\tau - \tau_c)^{-2/3} + \dots \end{aligned} \quad (86)$$

Therefore, the critical exponent α of \tilde{c}_v is $-2/3$ for black branes when $\tilde{d} \geq 2$.

Let us consider M5 as an example. We can expand (24) around the critical value as

$$\begin{aligned} b_q - b_c &= \frac{1}{3!} \left. \frac{\partial^3 b_q}{\partial x^3} \right|_{x=x_c} (x - x_c)^3 + \dots \\ &= -1.88444 (x - x_c)^3 + \dots \end{aligned} \quad (87)$$

$$\tilde{S} - \tilde{S}_c = 12.6268(x - x_c) + \dots. \quad (88)$$

We have now

$$\tau - \tau_c = 47.465(x - x_c)^3 + \dots, \quad (89)$$

$$\tilde{S} - \tilde{S}_c = 3.48741(\tau - \tau_c)^{1/3} + \dots, \quad (90)$$

and

$$\tilde{c}_v = 5.83415(\tau - \tau_c)^{-2/3} + \dots. \quad (91)$$

Note that the critical exponent for the specific heat has a universal value $-2/3$ as was also noted for the asymptotically AdS, dS and flat black holes earlier in [6, 8].

As indicated above already, the present analysis works for both the non-dilatonic and the dilatonic branes.

7 Conclusion

To conclude, in this paper we have studied in detail the equilibrium states and the phase structures of the asymptotically flat non-dilatonic and the dilatonic black branes in a cavity in arbitrary space-time dimensions D . Although we mostly concentrated on the non-dilatonic branes, the whole discussion applies also to the dilatonic branes as we give the details of how the analysis can be carried over in the Appendix. We considered only the canonical ensemble and so the charge inside the cavity and the temperature at the wall of the cavity were held fixed. We employed the Euclidean action formalism to compute the thermodynamics and the phase structure of the black branes. There is a marked difference in the phase structure when the charge enclosed in the cavity is zero and non-zero. When the charge is non-zero, there is also a qualitative difference in the phase structure when the $\tilde{d} > 2$, $\tilde{d} = 2$ and $\tilde{d} < 2$. We discussed them each separately.

For zero charge we found an analog of Hawking-Page transition even for these higher dimensional black objects in asymptotically flat background. So, we found for the zero charge case that there exists a minimum temperature given in eq.(36), below which there is no black brane phase and here the system will be in ‘hot flat space’ phase. But above this temperature there exists two black brane phase with different radii. The smaller one is unstable and the larger one is locally stable. When the temperature lies between the minimum value (36) and the value given in (42), the locally stable black brane will eventually tunnel into ‘hot flat space’ since the latter configuration in this region has lower free energy. There is a phase transition temperature given by (42) at which the black brane of size x_g given in (41) and the ‘hot flat space’ can coexist. But above this

temperature, the larger black brane becomes globally stable and the system can remain in this black brane phase. However, the smaller black brane still remains unstable. As we increase the temperature the size of the smaller black brane becomes smaller and that of the larger black brane becomes larger. As the temperature tends to infinity the size of the smaller black brane tends to zero and the larger black brane approaches the size of the cavity.

When the charge enclosed in the cavity is fixed but non-zero, we found that there exists a critical charge q_c at or above which there is a single globally stable black brane phase in the absence of an analog of ‘hot flat space’ or some other unknown configurations with favorable free energy for the charged system. When the charge is below this critical value, then there exists a certain range of temperature denoted by T_{\min} and T_{\max} below and above which there is a single globally stable black brane phase. But within this range there are three black brane phase with different radii. The largest and the smallest of which are locally stable as they correspond to the local minima of the free energy, but the intermediate one is unstable as it corresponds to the maximum of free energy. We found that the values of the free energies of the black brane phases depend on the temperature. In fact there exists a unique transition temperature at which the free energies of the largest and the smallest black brane become equal and so these two phases can coexist at this temperature and can make a phase transition freely from one phase to the other much like the van der Waals-Maxwell liquid gas phase transition. Above this transition temperature the larger black brane is globally stable and below this temperature the smaller black brane is globally stable. So, there is a first order phase transition from smaller black brane to larger black brane or *vice versa* above or below the transition temperature as the entropy of the system changes in this transition. In fact when the charge increases from $q < q_c$ towards $q = q_c$, there is a first order phase transition line which eventually ends up in a second order phase transition point (critical point) at $q = q_c$. At this critical point we have calculated the critical exponents and found that the critical exponent of specific heat has a universal value $-2/3$. We have elaborated this phase structure both analytically and numerically to illustrate the various situations. We found that this general phase structure is valid only for $\tilde{d} > 2$, where \tilde{d} is related to the dimensionality of the black p -brane by $\tilde{d} = D - p - 3$. $\tilde{d} = 2, 1$ case has been considered separately in subsections 5.1 and 5.2. $\tilde{d} = 1$ case is very similar in structure to the zero charge case except here b_q vs x curve starts from $x = q$ instead of $x = 0$ as in zero charge case. Also here there is no analog of ‘hot flat space’ since the system has non-zero charge. For $\tilde{d} = 2$, we found that there exists a critical charge q_c above which there is a single globally stable black brane phase when the temperature of the system is above a certain value, but below this

value of the temperature there is no black brane phase and we do not have a suitable description for this phase. When the charge becomes the critical value q_c , again there exists a temperature $T = \sqrt{3}/(2\pi\bar{\rho}_B)$ above which there is a single globally stable black brane phase and below this temperature there is no description available for the charged system. When the charge is below q_c , there is a certain range of temperature where there are two black brane phases, the smaller one is unstable and the larger one is globally stable. Above this range, there is a single globally stable black brane phase and below the minimum temperature we do not have a description available. This whole analysis works for both the non-dilatonic and dilatonic branes.

The only dimensionality which is relevant to the stability and phase structure is \tilde{d} and this implies that the branes related via the so-called double-dimensional reductions have the same stability and phase structure at least in the leading order approximation adopted. For example, this implies that the $D = 11$ M2 brane, the $D = 10$ fundamental string and the $D = 9$ 0-brane all have the same stability and phase structure, so do the $D = 11$ M5, the $D = 10$ D4, the $D = 9$ 3-brane, upto the $D = 6$ 0-brane, and so on. We also observe that the critical quantities (q_c, x_c, b_c) all decrease when \tilde{d} increases from 2 to 7.

Acknowledgements:

JXL would like to thank the participants of the advanced workshop “Dark Energy and Fundamental Theory”, supported by the Special Fund for Theoretical Physics from the NSF of China with grant no: 10947203, for stimulating discussions. He acknowledges support by grants from the Chinese Academy of Sciences, a grant from 973 Program with grant No: 2007CB815401 and a grant from the NSF of China with Grant No : 10975129.

Appendix

Here we will consider the dilatonic black p -brane solutions in D space-time dimensions. For studying thermodynamics we give their form in the Euclidean signature as,

$$\begin{aligned}
ds^2 &= \Delta_+ \Delta_-^{-\frac{d}{D-2}} dt^2 + \Delta_-^{\frac{\tilde{d}}{D-2}} \sum_{i=1}^{d-1} (dx^i)^2 + \Delta_+^{-1} \Delta_-^{\frac{a^2}{2\tilde{d}}-1} d\rho^2 + \rho^2 \Delta_-^{\frac{a^2}{2\tilde{d}}} \Omega_{\tilde{d}+1}^2, \\
A_{[p+1]} &= -ie^{a\phi_0/2} \left[\left(\frac{r_-}{r_+} \right)^{\tilde{d}/2} - \left(\frac{r_- r_+}{\rho^2} \right)^{\tilde{d}/2} \right] dt \wedge dx^1 \wedge \dots \wedge dx^p, \\
F_{[p+2]} &\equiv dA_{[p+1]} = -ie^{a\phi_0/2} \tilde{d} \frac{(r_- r_+)^{\tilde{d}/2}}{\rho^{\tilde{d}+1}} d\rho \wedge dt \wedge dx^1 \wedge \dots \wedge dx^p, \\
e^{2(\phi-\phi_0)} &= \Delta_-^a,
\end{aligned} \tag{92}$$

where Δ_{\pm} are as defined before in section 2. ϕ is the dilaton and ϕ_0 is its asymptotic value and related to the string coupling as $g_s = e^{\phi_0}$. a is the dilaton coupling given by¹¹,

$$a^2 = 4 - \frac{2d\tilde{d}}{D-2} \tag{93}$$

In the metric given in (92), the Euclidean time is periodic with periodicity β^* given as,

$$\beta^* = \frac{4\pi r_+}{\tilde{d}} \left(1 - \frac{r_-^{\tilde{d}}}{r_+^{\tilde{d}}} \right)^{\frac{1}{\tilde{d}} - \frac{1}{2}} \tag{94}$$

This is the inverse temperature at $\rho = \infty$. The local β is given as,

$$\beta = \Delta_+^{1/2} \Delta_-^{-\frac{d}{2(d+\tilde{d})}} \beta^* \tag{95}$$

which is the inverse of local temperature at ρ . The black p -brane will be placed in a cavity with its wall at $\rho = \rho_B$. It is clear from the metric in (92) that the physical radius of the cavity is

$$\bar{\rho}_B = \Delta_-^{\frac{a^2}{4\tilde{d}}} \rho_B, \tag{96}$$

while ρ_B is merely the coordinate radius. It is this $\bar{\rho}_B$ which we should fix in the following discussion and not ρ_B (as in the non-dilatonic case). Also we fix the dilaton on the boundary which is the requirement of obtaining its standard equation of motion from the action given later. In other words, we fix the dilaton at $\bar{\rho}_B$, which indicates that

¹¹Note that the form of a^2 is fixed by supersymmetry, in the sense that these are solutions of supergravity with maximal supersymmetry.

the asymptotic value of the dilaton is not fixed for the present consideration and this is crucial for our discussion. By this argument we also have

$$\bar{r}_{\pm} = \Delta_{-}^{\frac{a^2}{4d}} r_{\pm} \quad (97)$$

and \bar{r}_{\pm} are the proper parameters which we should use in the present context. In terms of the ‘barred’ variables Δ_{\pm} remain the same as before,

$$\Delta_{\pm} = 1 - \frac{r_{\pm}^{\tilde{d}}}{\rho_B^{\tilde{d}}} = 1 - \frac{\bar{r}_{\pm}^{\tilde{d}}}{\bar{\rho}_B^{\tilde{d}}} \quad (98)$$

For the canonical ensemble we have fixed local temperature at the wall of the cavity, fixed local brane volume $V_p = \Delta_{-}^{\frac{\tilde{d}(d-1)}{2(D-2)}} V_p^*$ and fixed charge defined as,

$$\begin{aligned} Q_d &= \frac{i}{2\sqrt{\kappa}} \int e^{-a(d)\phi} * F_{[p+2]} = \frac{\Omega_{\tilde{d}+1}}{2\sqrt{\kappa}} e^{-a\phi_0/2} \tilde{d}(r_+ r_-)^{\tilde{d}/2} \\ &= \frac{\Omega_{\tilde{d}+1} \tilde{d}}{\sqrt{2\kappa}} e^{-a\bar{\phi}/2} (\bar{r}_+ \bar{r}_-)^{\tilde{d}/2} \end{aligned} \quad (99)$$

where in the last line we have expressed the asymptotic value of the dilaton by the fixed dilaton $\bar{\phi} \equiv \phi(\bar{\rho}_B)$ at the wall of the cavity from the relation given in (92) and then expressed r_{\pm} by \bar{r}_{\pm} from (97).

With these data we will now evaluate the action. The relevant action for the gravity coupled to the dilaton and a $(p+1)$ -form gauge field with the Euclidean signature has the form

$$I_E = I_E(g) + I_E(\phi) + I_E(F) \quad (100)$$

where, $I_E(g)$ is the gravitational part of the action, $I_E(\phi)$ is the action for the dilaton and $I_E(F)$ is the action for the form-field and are given as,

$$\begin{aligned} I_E(g) &= -\frac{1}{2\kappa^2} \int_M d^D x \sqrt{g} R + \frac{1}{\kappa^2} \int_{\partial M} d^{D-1} x \sqrt{\gamma} (K - K_0), \\ I_E(\phi) &= -\frac{1}{2\kappa^2} \int_M d^D x \sqrt{g} \left(-\frac{1}{2} (\partial\phi)^2 \right), \\ I_E(F) &= \frac{1}{2\kappa^2} \frac{1}{2(d+1)!} \int_M d^D x \sqrt{g} e^{-a(d)\phi} F_{d+1}^2 \\ &\quad - \frac{1}{2\kappa^2} \frac{1}{d!} \int_{\partial M} d^{D-1} x \sqrt{\gamma} n_{\mu} e^{-a(d)\phi} F^{\mu\mu_1\mu_2\cdots\mu_d} A_{\mu_1\mu_2\cdots\mu_d}, \end{aligned} \quad (101)$$

The various quantities in the above actions have already been defined in section 2. The

equations of motion following from the action (100) have the forms,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}(\partial\phi)^2 g_{\mu\nu} + \frac{1}{2}\frac{1}{d!}e^{-a(d)\phi}\left(F_{\mu\mu_1\mu_2\cdots\mu_d}F_{\nu}{}^{\mu_1\mu_2\cdots\mu_d} - \frac{1}{2(d+1)}g_{\mu\nu}F^2\right) \quad (102)$$

$$\square\phi = -\frac{a(d)}{2(d+1)!}e^{-a(d)\phi}F_{[d+1]}^2 \quad (103)$$

$$\nabla_{\mu_1}(e^{-a(d)\phi}F^{\mu_1\cdots\mu_{d+1}}) = 0 \quad (104)$$

Using the equation of motion, the action can be reduced to:

$$I_E = \frac{d}{2(D-2)\kappa^2(d+1)!}\int_M d^Dx\sqrt{g}e^{-a(d)\phi}F_{d+1}^2 + \frac{1}{\kappa^2}\int_{\partial M} d^{D-1}x\sqrt{\gamma}(K - K_0) - \frac{1}{2\kappa^2}\frac{1}{d!}\int_{\partial M} d^{D-1}x\sqrt{\gamma}n_\mu e^{-a(d)\phi}F^{\mu\mu_1\mu_2\cdots\mu_d}A_{\mu_1\mu_2\cdots\mu_d}. \quad (105)$$

From the metric in (92) we have

$$n^\mu = \Delta_+^{1/2}\Delta_-^{\frac{1}{2}-\frac{a^2}{4d}}\delta_\rho^\mu, \quad (106)$$

The extrinsic curvature for the p -brane can be calculated as before,

$$K = -\nabla_\mu n^\mu = -\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}n^\mu) = -\Delta_+^{\frac{1}{2}}\Delta_-^{\frac{1}{2}-\frac{a^2}{4d}}\frac{1}{\rho}\left(1 - \frac{a^2}{4} + \frac{\tilde{d}}{2}\Delta_+^{-1} + \left(\frac{\tilde{d}}{2} + \frac{a^2}{4}\right)\Delta_-^{-1}\right) \quad (107)$$

The extrinsic curvature K_0 can be calculated as,

$$K_0 = -\frac{(\tilde{d}+1)\Delta_-^{-\frac{a^2}{4d}}}{\rho} = -\frac{\tilde{d}+1}{\bar{\rho}} \quad (108)$$

where we have defined $\bar{\rho} = \Delta_-^{\frac{a^2}{4d}}\rho$ and note that with this redefined ρ , K_0 takes exactly the same form as in the non-dilatonic case.

Now calculating each term in the action (105) separately as before we obtain,

$$I_E = -\frac{\beta V_p \Omega_{\tilde{d}+1}}{2\kappa^2}\bar{\rho}^{\tilde{d}}\left[2\left(\frac{\Delta_+}{\Delta_-}\right)^{1/2} + \tilde{d}\left(\frac{\Delta_-}{\Delta_+}\right)^{1/2} + \tilde{d}(\Delta_+\Delta_-)^{1/2} - 2(\tilde{d}+1)\right]. \quad (109)$$

Note that in the above action everything is expressed in terms of the ‘barred’ parameters defined earlier instead of the original parameters. Comparing the reduced action (109)

with the corresponding reduced action for the non-dilatonic branes (13) we find that they have exactly the same form in terms of the redefined parameters. Once we have this form of the action (109), we can rewrite it as before in the form $I_E = \beta E - S$ as,

$$I_E = -\frac{\beta V_p \Omega_{\tilde{d}+1} \bar{\rho}_B^{\tilde{d}}}{2\kappa^2} \left[(\tilde{d}+2) \left(\frac{\Delta_+}{\Delta_-} \right)^{1/2} + \tilde{d}(\Delta_+ \Delta_-)^{1/2} - 2(\tilde{d}+1) \right] \Big|_{\bar{\rho}=\bar{\rho}_B} - \frac{4\pi V_p^* \Omega_{\tilde{d}+1}}{2\kappa^2} r_+^{\tilde{d}+1} \left(1 - \frac{r_-^{\tilde{d}}}{r_+^{\tilde{d}}} \right)^{\frac{1}{2}+\frac{1}{\tilde{d}}}, \quad (110)$$

where we have used

$$\beta^* = \frac{4\pi r_+}{\tilde{d}} \left(1 - \frac{r_-^{\tilde{d}}}{r_+^{\tilde{d}}} \right)^{\frac{1}{\tilde{d}}-\frac{1}{2}}, \quad (111)$$

We can thus identify the entropy

$$S = \frac{4\pi V_p^* \Omega_{\tilde{d}+1}}{2\kappa^2} r_+^{\tilde{d}+1} \left(1 - \frac{r_-^{\tilde{d}}}{r_+^{\tilde{d}}} \right)^{\frac{1}{2}+\frac{1}{\tilde{d}}} \quad (112)$$

and the energy of the cavity as,

$$E = -\frac{V_p \Omega_{\tilde{d}+1}}{2\kappa^2} \bar{\rho}_B^{\tilde{d}} \left[(\tilde{d}+2) \left(\frac{\Delta_+}{\Delta_-} \right)^{1/2} + \tilde{d}(\Delta_+ \Delta_-)^{1/2} - 2(\tilde{d}+1) \right] \quad (113)$$

Note that the entropy has exactly the same form as that of the non-dilatonic brane and we find that the energy approaches the ADM mass at $\bar{\rho}_B \rightarrow \infty$ as expected. We would like to remark that all the quantities like energy, entropy and temperature in the cavity all have the invariant forms in terms of ‘unbarred’ (non-dilatonic case) and ‘barred’ (dilatonic case) coordinates. Energy expression given in (113) has already the same form as can be compared with (16). Entropy given in (112) can be written as,

$$\begin{aligned} S &= \frac{4\pi V_p^* \Omega_{\tilde{d}+1}}{2\kappa^2} r_+^{\tilde{d}+1} \left(1 - \frac{r_-^{\tilde{d}}}{r_+^{\tilde{d}}} \right)^{\frac{1}{2}+\frac{1}{\tilde{d}}} \\ &= \frac{4\pi V_p \Omega_{\tilde{d}+1}}{2\kappa^2} \bar{r}_+^{\tilde{d}+1} \Delta_-^{-\frac{1}{2}-\frac{1}{\tilde{d}}} \left(1 - \frac{\bar{r}_-^{\tilde{d}}}{\bar{r}_+^{\tilde{d}}} \right)^{\frac{1}{2}+\frac{1}{\tilde{d}}} \end{aligned} \quad (114)$$

Comparing (114) with (15) we find that indeed they have exactly the same form. Similarly we have from (95) and (94) the expression for inverse temperature as,

$$\beta = \frac{4\pi \bar{r}_+}{\tilde{d}} \Delta_+^{\frac{1}{2}} \Delta_-^{-\frac{1}{\tilde{d}}} \left(1 - \frac{\bar{r}_-^{\tilde{d}}}{\bar{r}_+^{\tilde{d}}} \right)^{\frac{1}{\tilde{d}}-\frac{1}{2}} \quad (115)$$

Comparing (115) with (21) we again find that they have exactly the same form. This therefore indicates that the thermodynamical quantities of the non-dilatonic branes essentially have the same structure as the dilatonic branes with the new physical parameters. Now using the expression of charge (99) we can write \bar{r}_- in terms of \bar{r}_+ as,

$$\bar{r}_- = \left(\frac{\sqrt{2}\kappa Q_d}{\Omega_{\tilde{d}+1}\tilde{d}} e^{a\bar{\phi}/2} \right)^{\frac{2}{\tilde{d}}} \frac{1}{\bar{r}_+} = \frac{(Q_d^*)^2}{\bar{r}_+} \quad (116)$$

Where we have defined $Q_d^* = [(\sqrt{2}\kappa Q_d e^{a\bar{\phi}/2})/(\Omega_{\tilde{d}+1}\tilde{d})]^{1/\tilde{d}}$. Note that since the charge Q_d is fixed inside the cavity and the dilaton $\bar{\phi}$ is fixed at the wall of the cavity Q_d^* is also fixed. Therefore, \bar{r}_- is not an independent parameter, but is dependent on \bar{r}_+ as given in (116). Now using (116) we can write (115) as,

$$\beta = \frac{4\pi\bar{r}_+}{\tilde{d}} \left(1 - \frac{Q_d^{*2\tilde{d}}}{\bar{r}_+^{2\tilde{d}}} \right)^{\frac{1}{\tilde{d}} - \frac{1}{2}} \left(1 - \frac{\bar{r}_+^{\tilde{d}}}{\bar{\rho}_B^{\tilde{d}}} \right)^{1/2} \left(1 - \frac{Q_d^{*2\tilde{d}}}{\bar{r}_+^{\tilde{d}}\bar{\rho}_B^{\tilde{d}}} \right)^{\frac{1}{\tilde{d}}}, \quad (117)$$

Now as before we can define

$$x = \left(\frac{\bar{r}_+}{\bar{\rho}_B} \right)^{\tilde{d}} \leq 1, \quad \bar{b} = \frac{\beta}{4\pi\bar{\rho}_B}, \quad q = \left(\frac{Q_d^*}{\bar{\rho}_B} \right)^{\tilde{d}} \quad (118)$$

In terms of these parameters (117) takes the form,

$$\bar{b} = \frac{1}{\tilde{d}} x^{1/\tilde{d}} (1-x)^{1/2} \left(1 - \frac{q^2}{x^2} \right)^{\frac{1}{\tilde{d}} - \frac{1}{2}} \left(1 - \frac{q^2}{x} \right)^{-\frac{1}{\tilde{d}}} \equiv b_q(x) \quad (119)$$

Comparing with (24) and (25) we find that this equation (119) has exactly the same form as for the non-dilatonic branes. The expression of $b_q(x)$ was crucial for our analysis for the equilibria and stability structure of the black p -branes. Since the dilatonic branes have the same expression for $b_q(x)$ as the non-dilatonic branes, the phase structure for the dilatonic branes would be exactly the same as the non-dilatonic branes. The reduced Euclidean action \tilde{I}_E can be seen from (110) to take exactly the same form as the non-dilatonic branes given in (27), (28) and (29) in terms of the new parameters (118).

So far, we have not addressed the issue regarding the validity of using the effective action in describing the phase structure of black p -branes throughout the parameter space considered¹². Here we will address this for both non-dilatonic and dilatonic branes together. For non-dilatonic branes, we need to keep the curvature of black brane spacetime uniformly weak throughout the parameter space. For dilatonic branes, in addition, we

¹²We thank the anonymous referee for raising this concern.

also need to keep the effective string coupling uniformly weak. The effective string coupling can be read from (92) and the curvature can be calculated from the metric given in the same equation as

$$\begin{aligned} e^\phi &= g_s \Delta_-^{a/2}, \\ R &= \frac{\tilde{d}^2}{2} \left[\frac{a^2}{4} \frac{\Delta_+}{\Delta_-} \left(\frac{\bar{r}_-}{\bar{\rho}} \right)^{2\tilde{d}} \frac{\bar{r}_+ \bar{r}_-}{\bar{\rho}^2} + \frac{d - \tilde{d}}{D - 2} \left(\frac{\bar{r}_+ \bar{r}_-}{\bar{\rho}^2} \right)^{\tilde{d}+1} \right] \frac{1}{\bar{r}_- \bar{r}_+}, \end{aligned} \quad (120)$$

where $g_s = e^{\phi_0}$ is the asymptotic string coupling. Note that for the scalar curvature each term in the square bracket is less than unity since $\bar{\rho} \geq \bar{r}_+ > \bar{r}_-$, $\Delta_+/\Delta_- < 1$, $a^2/4 < 1$ and $(d - \tilde{d})/(D - 2) < 1$. In other words, the square bracket contributes at most a factor of order unity to the curvature. So in order to keep the curvature uniformly weak, we need to have

$$l^2 R \sim \frac{l^2}{\bar{r}_- \bar{r}_+} = l^2 \left[\frac{\Omega_{\tilde{d}+1}}{\sqrt{2} \kappa Q_d e^{a\phi_0/2} \Delta_-^{a^2/4}} \right]^{2/\tilde{d}} \ll 1, \quad (121)$$

where l is the relevant length scale under consideration, for example, it is the Planck scale l_p in eleven dimensions or the string scale l_s in ten dimensions. Note that the charge quantization gives $\sqrt{2} \kappa Q_d e^{a\phi_0/2} / \Omega_{\tilde{d}+1} \sim N l^{\tilde{d}}$ with the integer N labeling the number of branes. So the uniformly weak curvature condition is

$$N \Delta_-^{a^2/4} \gg 1. \quad (122)$$

For M branes, $a = 0$ and a weak curvature is the only requirement which can be satisfied when $N \gg 1$. For D3 branes, we need in addition a small g_s . For those branes, it doesn't appear that there is any constraint on the parameter space we considered. For dilatonic branes, the condition (122) for weak curvature can easily be satisfied for non-extremal branes, i.e., $r_+ > r_-$, for given large enough N since Δ_- is finite for $\rho \geq r_+$. Now the effective string coupling as given in (120) can remain small for small g_s when $a > 0$ and can also remain small if g_s is chosen to be small enough for $a < 0$. For this case, the parameter x used in the text falls in the range $q < x \leq 1$. If we consider extremal branes, i.e., $r_+ = r_-$, we then have to limit to the range $\rho > r_+ = r_-$ so that the curvature remains small. This can also give a small effective string coupling even when $a < 0$ by the same token. Now we have the range of parameter space as $q \leq x < 1$. In other words, for dilatonic branes, we can consider the edge state only at one end, i.e., either at $x = q$ or at $x = 1$, but not at both.

The discussion presented in this paper corresponds to the single-scalar $\Delta = 4$ black branes given in [12] (do not confuse this Δ with the discriminant notation used in the

main text of this paper). However, we have investigated the other $\Delta = 3, 2, 1$ single scalar black branes discussed in [12] as well and found that the basic phase structure remains the same as that of $\Delta = 4$ black branes even though the minimal value of \tilde{d} at which there exists a critical charge q_c depends on the value of the respective Δ .

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